EULER'S RATIO-SUM THEOREM

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Abstract. In this paper we present two proofs of Euler's ratio-sum theorem for the concurrent cevians of a planar triangle and provide an analogue of the theorem and its proof for spherical triangles.

1. Introduction

Two well-known theorems in triangle geometry relating to concurrency and collinearity respectively are Ceva's theorem, which provides a relationship between the parts of the sides of a triangle that are created by drawing concurrent cevians, and Menelaus' theorem, which describes the relationship between the ratios formed by drawing a transversal line that cuts the three sides of the triangle. Euler's ratio-sum theorem (check [1]) is of a similar type: it involves a relation between the parts of a triangle's concurrent cevians that can then be used to construct the triangle given the lengths of those parts.

2. The Triangle Construction Problem

Around 1812, Euler considered the problem of constructing a triangle when the lengths of the 'six parts' of its concurrent cevians (considering that they are divided by the point of intersection) are given. He soon noticed a relation between the six lengths which he used to solve the problem. His proof of this theorem offers another glance into his masterful command of algebraic manipulation, relying only on standard formulae for the area of a triangle and the angle-sum property of the triangle.

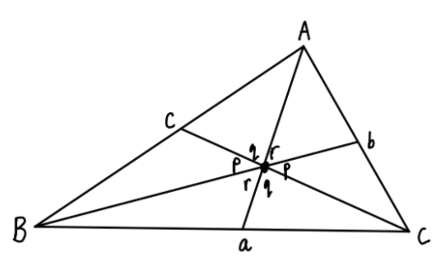

Figure 1

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Theorem 2.1. *Consider a triangle ABC and cevians Aa, Bb and Cc that coincide at O. Then the following property always holds:*

$$\frac{AO}{Oa} \cdot \frac{BO}{Ob} \cdot \frac{CO}{Oc} = \frac{AO}{Oa} + \frac{BO}{Ob} + \frac{CO}{Oc} + 2.$$

Remark 2.2. Notice that if we take O to be the centroid of the triangle, the ratios $\frac{AO}{Oa}$, $\frac{BO}{Ob}$ and $\frac{CO}{Oc}$ are all 2. In that case, the right-hand side of our equation becomes $2^3 = 8$ and the left-hand side is $2(3) + 2 = 8$.

Proof. We can assign names to the parts AO, BO, CO and Oa, Ob, Oc as follows:

$$AO = A, BO = B, CO = C, Oa = a, OB = b, OC = c.$$

From Figure 1, we can see that $p + q + r = 180°$. We now turn to the popular technique of using triangle area ratios to establish relations between the sides and angles of the triangle. We have,

$$[AOc] = \frac{1}{2} Ac \sin q.$$

Similarly, $[BOc] = \frac{1}{2} Bc \sin p$, and $[AOB] = \frac{1}{2} AB \sin(p + q) = \frac{1}{2} AB \sin r$.

Remark 2.3. Note that the last equality followed from the identity $\sin x = \sin(180 - x)$.

Now, since $[AOB] = [AOc] + [BOc]$, we have

$$AB \sin r = Ac \sin q + Bc \sin p. \tag{2.1}$$

We can similarly use area ratios on triangles $[AOC]$ and $[BOC]$ to obtain the equations

$$BC \sin p = Ba \sin c + Ca \sin q. \tag{2.2}$$

$$CA \sin q = Cb \sin p + Ab \sin r. \tag{2.3}$$

We divide both sides of Equation 2.1 by ABc to obtain

$$\frac{\sin r}{c} = \frac{\sin q}{B} + \frac{\sin p}{A} \tag{2.4}$$

We divide Equation 2.2 and Equation 2.3 by BCa and CAb respectively to obtain

$$\frac{\sin p}{a} = \frac{\sin r}{C} + \frac{\sin q}{B}.$$

$$\frac{\sin q}{b} = \frac{\sin p}{A} + \frac{\sin r}{C}.$$

We can now set $A = \alpha a$, $B = \beta b$ and $C = \gamma c$. To avoid having to write out long and confusing equations involving several variables, we set

$$\frac{\sin p}{A} = \frac{\sin p}{\alpha a} = P. \tag{2.5}$$

Similarly, we define Q and R as follows:

$$\frac{\sin q}{B} = \frac{\sin q}{\beta b} = Q.$$
$$\frac{\sin r}{C} = \frac{\sin r}{\gamma c} = R.$$

Combining Equation 2.4 with Equation 2.5, we get

$$\gamma R = P + Q. \tag{2.6}$$

Analogously, $\alpha P = Q + R$ and $\beta Q = R + P$.

Taking the pairwise differences of these equations and rearranging, we get

$$\frac{P}{R} = \frac{\gamma + 1}{\alpha + 1}$$
$$\frac{Q}{P} = \frac{\alpha + 1}{\beta + 1}$$
$$\frac{R}{Q} = \frac{\beta + 1}{\gamma + 1}.$$

From these equations, we get that

$$P : Q : R = \frac{1}{\alpha + 1} : \frac{1}{\beta + 1} : \frac{1}{\gamma + 1}.$$

Thus, we have established a relationship between P, Q, R and α, β, γ. Combining Equation 2.6 and Equation 2.7 we have

$$\frac{P}{Q} = \frac{\gamma + 1}{a\gamma - 1}.$$

Furthermore, since $\frac{P}{Q} = \frac{\beta+1}{\alpha+1}$, we obtain

$$\alpha\beta\gamma = \alpha + \beta + \gamma + 2.$$

Now,

$$\alpha = \frac{AO}{Oa}, \quad \beta = \frac{BO}{Ob}, \quad \gamma = \frac{CO}{Oc}.$$

Thus, replacing α, β and γ with their corresponding side ratios, we have established the theorem. $\square$

Now that we have established a relationship between the 'six parts' of the cevians, we return to the Triangle Construction problem. We retain the notation we introduced while proving the previous theorem. Specifically, we have

$$A = \alpha a, B = \beta b, C = \gamma c$$

and

$$\sin p = \alpha a P, \sin q = \beta b Q, \sin r = \gamma c R.$$

By the previous theorem, $\alpha\beta\gamma = \alpha + \beta + \gamma + 2$. Now, we use the ratio $P : Q : R$ to set

$$P = \frac{\delta}{\alpha + 1}, \quad Q = \frac{\delta}{\beta + 1}, \quad R = \frac{\delta}{\gamma + 1}.$$

Here, δ is introduced to reduce the amount of variables we need to deal with later. Once we find its value, we will have all the information we need about the angles and line segments to construct the triangle. Until now, most of our expressions pertain to the sines of the angles p, q and r, so it makes sense to work with these as we proceed. Again, we employ the technique of introducing new variables to avoid clutter. We set

$$\sin p = f, \quad \sin q = g, \quad \sin r = h.$$

Now, since we have $\sin r = \sin(p + q) = \sin p \cos q + \cos p \sin q$ and $\cos q = \sqrt{1 - (\sin q)^2}$ we can write

$$h = f\sqrt{1 - g^2} + g\sqrt{1 - f^2}.$$

Squaring, we get

$$h^2 = f^2 + g^2 - 2f^2g^2 + 2fg\sqrt{(1 - f^2)(1 - g^2)}.$$

Since we want to remove the radicals completely, we square again to get

$$f^4 + g^4 + h^4 - 2f^2g^2 - 2f^2h^2 - 2g^2h^2 + 4f^2g^2h^2 = 0. \tag{2.7}$$

Since $f = \sin p$, by Equation 2.5, we have

$$f = \alpha a P = \frac{\alpha a \delta}{\alpha + 1}.$$

Similarly, $g = \dfrac{\beta b \delta}{\beta + 1}$, and $h = \dfrac{\gamma c \delta}{\gamma + 1}$. By another application of our favourite technique of introducing new variables to simplify calculations, we set

$$\frac{\alpha a}{\alpha + 1} = F, \ \ \frac{\beta b}{\beta + 1} = G, \ \ \frac{\gamma c}{\gamma + 1} = H. \tag{2.8}$$

This means that $f = F\delta$, $g = G\delta$, and $h = H\delta$. The advantage of doing this, as we shall soon see, is that we can directly solve for δ. We have, by Equation 2.7,

$$\delta^4\left(F^4+G^4+H^4-2F^2G^2-2F^2H^2-2G^2H^2+4\delta^2F^2G^2H^2\right)=0.$$
$$F^4+G^4+H^4-2F^2G^2-2F^2H^2-2G^2H^2+4\delta^2F^2G^2H^2=0.$$

Thus

$$\delta^2=\frac{2F^2G^2+2F^2H^2+2G^2H^2-F^4-G^4-H^4}{4F^2G^2H^2}.$$

Taking square roots and factoring,

$$\delta=\frac{\sqrt{(F+G+H)(F+G-H)(F+H-G)(G+H-F)}}{2FGH}.$$

This expression looks similar to another expression we are already familiar with. Heron's formula for the area M of a triangle with side lengths a_1, a_2, a_3 assumes a similar form:

$$M=\sqrt{s(s-a_1)(s-a_2)(s-a_3)}$$

where $s=\dfrac{a_1+a_2+a_3}{2}$.

Considering a triangle with side lengths F, G and H, the area M of the triangle by Heron's formula and some algebraic manipulation that suits our purposes is

$$M=\frac{1}{4}\sqrt{(F+G+H)(F+G-H)(F+H-G)(G+H-F)}.$$

This looks similar to the expression we obtained for δ. Comparing, we get

$$\delta=\frac{2M}{FGH}$$

Recall that F, G and H can be found using just the 'six parts' of our cevians (see Equation 2.8). Also, M, as the area of the triangle with sides F, G and H can naturally be computed using just those values. Thus, δ can be computed using the information we have already been presented with.

Recall that we had defined f, g and h as the sines of angles p, q and r respectively. Now, since $f=\dfrac{\alpha a\delta}{\alpha+1}$, $g=\dfrac{\beta b\delta}{\beta+1}$, and $h=\dfrac{\gamma c\delta}{\gamma+1}$, knowing the value for δ will give us the value for the sines of the angles. Since we have the constraint $p+q+r=180°$, the sines of the angles will give only a few possibilities for the measure of the angles. Notice also that knowing the measure of one of the angles will give us the measure of the other two. Once we know the angles, we can construct the triangles $\Delta BOc, \Delta cOA, \Delta AOb, \Delta bOC, \Delta COa$ and ΔaOb from Figure 1 using the lengths we have been given. Thus, ΔABC can be easily constructed.

As Euler explored this problem, he noticed that Theorem 2.1 can be stated more elegantly as follows.

Theorem 2.4. *Consider the triangle ΔABC and cevians Aa, Bb and Cc that coincide at O (check Figure 1). Then, the following property always holds:*

$$\frac{Oa}{Aa} + \frac{Ob}{Bb} + \frac{Oc}{Cc} = 1.$$

Proof. Once again, we retain our original notation to set $AO = \alpha \cdot Oa$, $BO = \beta \cdot Ob$, and $CO = \gamma \cdot OC$. By Theorem 2.1, we have $\alpha\beta\gamma = \alpha + \beta + \gamma + 2$. We add

$$\alpha\beta + \alpha\gamma + \beta\gamma + \alpha + \beta + \gamma + 1$$

to both sides to get an expression that lends itself well to factorisation. Adding and factoring, we get

$$(\alpha + 1)(\beta + 1)(\gamma + 1) = (\alpha + 1)(\beta + 1) + (\alpha + 1)(\gamma + 1) + (\beta + 1)(\gamma + 1).$$

Dividing by $(\alpha + 1)(\beta + 1)(\gamma + 1)$ on both sides yields

$$\frac{1}{\gamma + 1} + \frac{1}{\beta + 1} + \frac{1}{\alpha + 1} = 1.$$

Notice that we had defined α, β and γ to be the ratios $\frac{AO}{Oa}, \frac{BO}{Ob}$ and $\frac{CO}{Oc}$ respectively. Substituting these values into our expression will yield the desired equality.

Further, since we have

$$\frac{\gamma + 1}{\gamma + 1} + \frac{\beta + 1}{\beta + 1} + \frac{\alpha + 1}{\alpha + 1} = 3$$

we can take the difference of the two expressions to obtain

$$\frac{\alpha}{\alpha + 1} + \frac{\beta}{\beta + 1} + \frac{\gamma}{\gamma + 1} = 2. \tag{2.9}$$

The above expression may prove to be more useful than its equivalent form in some cases. □

3. An alternate proof and some extensions

Euler offered another, arguably simpler proof of the previous theorem that relies on the clever addition of segments to our given triangle and then using properties of similar triangles. Since our previous proof relied on long-winded algebraic techniques involving the introduction of several new variables, we have included this proof here to simplify matters.

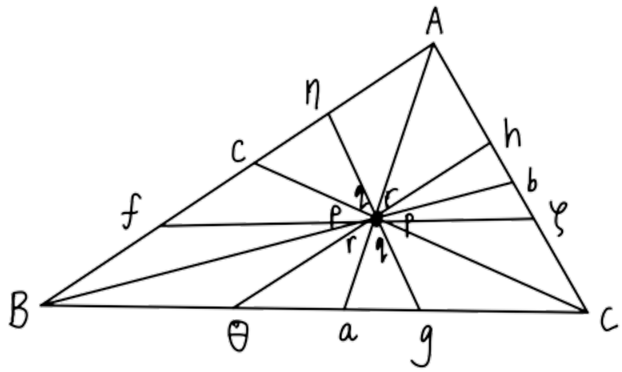

Figure 2

We draw segments $f\zeta$, $g\eta$ and $h\theta$ through O parallel to BC, AC and AB respectively. Using the previous theorem, we have

$$\frac{Bf}{AB} + \frac{A\eta}{AB} + \frac{f\eta}{AB} = 1.$$

Now, since $\triangle Aba$ and $\triangle AfO$ are similar,

$$\frac{Bf}{BA} = \frac{Oa}{Aa}.$$

The first fraction $\frac{Bf}{AB}$ can be represented as $\frac{Oa}{Aa}$. Using the similarity relations $\triangle BAb \sim \triangle B\eta O$ and $\triangle fO\eta \sim \triangle BCA$, we get the following

$$\frac{A\eta}{AB} = \frac{Ob}{Bb}$$
$$\frac{f\eta}{BA} = \frac{fO}{BC}.$$

Now, $fO = B\theta$, so, since $\triangle BCc \sim \triangle \theta CO$, the fraction $\frac{B\theta}{BC}$ will be $\frac{Oc}{Cc}$. the identity $\frac{Bf+A\eta+f\eta}{AB} = 1$ assumes the form:

$$\frac{Oa}{Aa} + \frac{Ob}{Bb} + \frac{Oc}{Cc} = 1,$$

Which is what we had set out to prove.

This property also holds when the point O is taken anywhere outside the triangle, as in Figure 3. For this case, we set $AO = A$ and $Oa = a$. Also, we have $BO = b$. Since O lies on the boundary of the triangle and b lies in its interior, we set $Ob = -b$, contrary to our choice

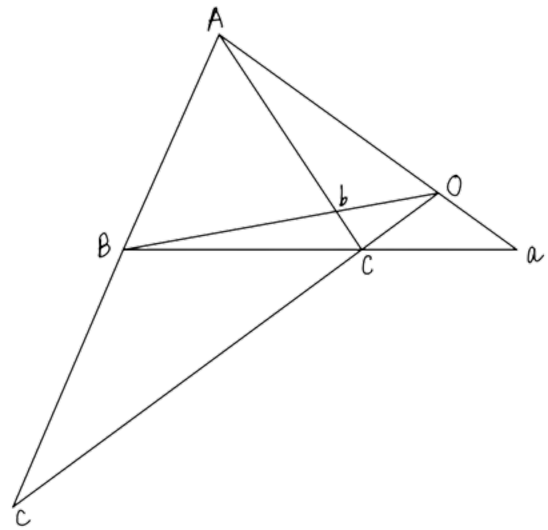

Figure 3

of $Ob = b$ for the previous cases. Similarly, we have $CO = C$ and $Oc = -c$. It is apparent that

$$Aa = A + a$$
$$Bb = B + b$$
$$Cc = -(C + c).$$

Therefore, since we always have

$$\frac{a}{a+A} + \frac{b}{b+B} + \frac{c}{c+C} = 1,$$

for the lines drawn in Figure 3 we will have

$$\frac{Oa}{Aa} - \frac{Ob}{Bb} + \frac{Oc}{Cc} = 1.$$

The property $\frac{\alpha}{\alpha+1} + \frac{\beta}{\beta+1} + \frac{\gamma}{\gamma+1} = 2$ (Check Equation 2.9) also helps us to find the area of the whole triangle ABC. We have

$$[AOB] = \frac{1}{2}AB\sin(p+q) = \frac{1}{2}AB\sin r.$$

Recall that we have $\sin r = CR$, which yields

$$[AOB] = \frac{1}{2}ABCR.$$

Similarly, we have $[AOC] = \dfrac{1}{2}ABCQ$ and $[BOC] = \dfrac{1}{2}ABCP$. Now, since

$[ABC] = [AOB] + [AOC] + [BOC]$ we have

$$[ABC] = \frac{1}{2}ABC(P + Q + R).$$

Retaining the notation from the previous sections, we make the following substitutions:

$$P = \frac{F\delta}{A}, \ \ Q = \frac{G\delta}{B}, \ \ R = \frac{H\delta}{C}.$$

Recall that we had $F = \frac{A}{\alpha+1}$, $G = \frac{B}{\beta+1}$ and $C = \frac{C}{\gamma+1}$. Thus, the area of the triangle becomes

$$[ABC] = \frac{1}{2}ABC\delta\left(\frac{1}{\alpha+1} + \frac{1}{\beta+1} + \frac{1}{\gamma+1}\right).$$

From our previous work we know that $\dfrac{1}{\alpha+1} + \dfrac{1}{\beta+1} + \dfrac{1}{\gamma+1} = 1$ so that

$$[ABC] = \frac{1}{2}ABC\delta$$

.

Now, since we had set the area of the triangle with sides F, G and H equal to M and found that $\delta = \frac{2M}{FGH}$, the area of our triangle ABC can be expressed as $\frac{ABCM}{FGH}$.

By Equation 2.8 we can substitute the values of F, G and H in the expression for the area of the triangle ABC to find that

$$[ABC] = (\alpha + 1)(\beta + 1)(\gamma + 1)M.$$

Thus we have found an an easier way to derive the property in Theorem 2.4 and have found a counterpart of the theorem for the case when the point O lies on the boundary of the triangle. We also used our previous results to find a convenient expression for the area of the triangle ABC for the case where the point O lies inside the triangle.

4. Ratio-sum theorem for Spherical Triangles

We now present a counterpart of the ratio-sum theorem for spherical triangles, where the sum of the angles is greater than 180°.

Definition 4.1. A *spherical triangle* is a figure formed on the surface of a sphere by three great circular arcs intersecting pairwise in three vertices.

The results concerning triangles in the plane can be modified for spherical triangles. We consider a spherical triangle ABC (Figure 4), in which arcs from each vertex are drawn to the opposite sides so as to coincide at a point O. We find that the following relation holds

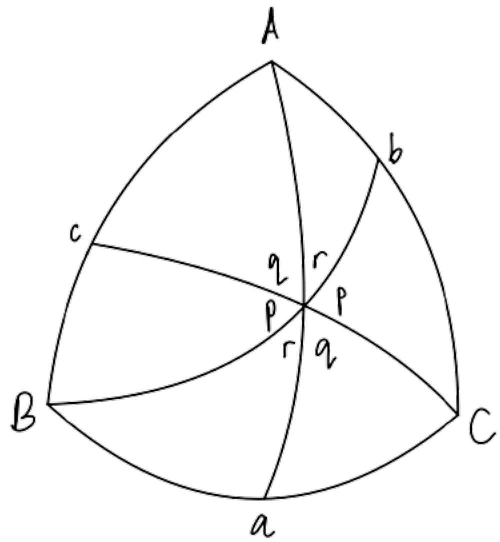

Figure 4

between the 'six parts' of the cevians, so that, given these values, we can construct the triangle.

Theorem 4.2. *If, in a spherical triangle ABC, arcs Aa, Bb, and Cc are drawn from each vertex to its opposite side so as to coincide at a point O, and if we set*

$$\frac{\tan AO}{\tan Oa} = \alpha, \quad \frac{\tan BO}{\tan Ob} = \beta, \quad \frac{\tan CO}{\tan Oc} = \gamma,$$

then we will always have $\alpha\beta\gamma = \alpha + \beta + \gamma + 2$, which may be simplified as

$$\frac{1}{\alpha+1} + \frac{1}{\beta+1} + \frac{1}{\gamma+1} = 1.$$

Remark 4.3. Note that this has the same form as Theorem 2.1.

Proof. Let the angles around the point of intersection be named as marked in the figure, and also set arcs $AO = A$, $BO = B$, $CO = C$, $Oa = a$, $Ob = b$, $Oc = c$. For a spherical triangle ABC with opposite sides a, b and c we have

$$\tan A = \frac{\sin a \sin C}{\cos a \sin b - \sin a \cos b \cos C}.$$

Thus, using the formula for the tangent of $\angle AOc$ we get

$$\tan \angle AcO = \frac{\sin A \sin q}{\cos A \sin c - \sin A \cos c \cos q},$$

and in triangle BOc we will have

$$\tan \angle BcO = \frac{\sin B \sin p}{\cos B \sin c - \sin B \cos c \cos p}.$$

Now since these two angles when summed together make two right angles, the sum of their tangents should equal zero. Adding the two equations, we have:

$$\sin A \cos B \sin c \sin q - \sin A \sin B \cos c \cos p \sin q + \sin B \cos A \sin c \sin p - \sin A \sin B \cos c \cos q \sin p = 0.$$

This can be reduced to:

$$\sin A \cos B \sin c \sin q + \sin B \cos A \sin c \sin p = \sin A \sin B \cos c \sin r,$$

from which we gather

$$\sin r = \frac{\sin A \cos B \sin c \sin q + \sin B \cos A \sin c \sin p}{\sin A \sin B \cos c}.$$

which in turn produces the following equation:

$$\frac{\sin r}{\tan c} = \frac{\sin p}{\tan A} + \frac{\sin q}{\tan B}.$$

In the same way, we will have:

$$\frac{\sin p}{\tan a} = \frac{\sin q}{\tan B} + \frac{\sin r}{\tan C},$$

$$\frac{\sin q}{\tan b} = \frac{\sin r}{\tan C} + \frac{\sin p}{\tan A}.$$

Moreover, since in the beginning we set

$$\frac{\tan A}{\tan a} = \alpha, \quad \frac{\tan B}{\tan b} = \beta, \quad \frac{\tan C}{\tan c} = \gamma,$$

when these values are substituted, we get:

$$\frac{\sin r}{\tan c} = \frac{\sin p}{\alpha \tan a} + \frac{\sin q}{\beta \tan b},$$

$$\frac{\sin p}{\tan a} = \frac{\sin q}{\beta \tan b} + \frac{\sin r}{\gamma \tan c},$$

$$\frac{\sin q}{\tan b} = \frac{\sin r}{\gamma \tan c} + \frac{\sin p}{\alpha \tan a}.$$

Let us now further set

$$\frac{\sin p}{\alpha \tan a} = P, \quad \frac{\sin q}{\beta \tan b} = Q, \quad \frac{\sin r}{\gamma \tan c} = R.$$

Once this is done, our three equations will be

$$\gamma R = P + Q, \quad \alpha P = Q + R, \quad \beta Q = R + P.$$

The first of these becomes $R = \frac{P+Q}{\gamma}$, and the second becomes $R = \alpha P - Q$. Setting them equal, we get $\frac{P}{Q} = \frac{\gamma+1}{\alpha\gamma-1}$ Subtracting the third equation from the second gives $\alpha P - \beta Q = Q - P$, and we deduce that $\frac{P}{Q} = \frac{\beta+1}{\alpha+1}$. Finally,

$$\frac{\gamma+1}{\alpha\gamma-1} = \frac{\beta+1}{\alpha+1},$$

Simplifying, we get the desired expression

$$\alpha\beta\gamma = \alpha + \beta + \gamma + 2.$$

Notice that the proof of the ratio-sum theorem for spherical triangles is very similar to the proof of the theorem for planar triangles. We set our intermediate ratios equal to some P, Q and R for the sake of brevity, and substitute the original values at the final stage to obtain the desired expression.

□

5. Acknowledgements

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References

[1] Euler, L. (1812). *Geometrica et sphaerica quedam.* Retrieved from `http://eulerarchive.maa.org//docs/originals/E749.pdf`